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DIFFUSION ON A PARTICLE IN THE SHEAR FLOW OF A VISCOUS FLUID. APPROXIMATION OF THE DIFFUSION BOUNDARY LAYER

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The steady convective diffusion on the surface of a particle of a substance dissolved in a uniform shear flow of viscous flow is considered. The problem of diffusion on a solid sphere and a spherical drop is solved in the approximation of the diffusion boundary layer.

Determination of diffusion afflux of a substance (or heat) on the surface of a moving particle is one of the fundamental problems of physicochemical hydrodynamics related to the theory of combustion, chemical reactors, in particular those with suspended layers, to the theory of coagulation and flocculation of disperse systems, deposition of aerosols, and in numerous other applications.

The analytical solutions obtained so far relate only to straight, uniform at infinity, laminar flows past particles at low Reynolds numbers [1 - 6].

Here an approximate analytical expression is derived for the diffusing stream of a substance on the surface of a spherical particle in a uniform laminar shear flow. Stokes' approximation derived in [7] is used for determining the shear flow field. It is assumed that the Péclet number is considerable so that the equation of convective diffusion can be expressed in terms of boundary layer approximation.

1. Statement of problem. Let us consider a spherical particle carried along by a stream of viscous incompressible fluid in a steady uniform shear flow. In an orthogonal Cartesian system of coordinates attached to the particle the velocity distribution of the unperturbed flow (at great distances from the particle) is a linear function of coordinates

$$\mathbf{v}_0 = (\boldsymbol{\alpha}\mathbf{r}) \tag{1.1}$$

where $\mathbf{r}$ is the radius vector and $\boldsymbol{\alpha}$ is the constant symmetric second-rank tensor which, without loss of generality, may be written in the form

$$\boldsymbol{\alpha} = \begin{Vmatrix} -\alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 2\alpha \end{Vmatrix} \tag{1.2}$$

For the flow field around the sphere corresponding to the conditions at infinity (1.1) and (1.2) the formula derived by Einstein for a solid particle in which the inertial terms in the Navier-Stokes equations have been neglected, and extended by Taylor [7] to the case of a drop. The stream function in spherical coordinates is of the form

$$\psi = \alpha a^3 \left(\frac{r^3}{a^3} - \frac{5}{2} M_1 + \frac{3}{2} M_2 \frac{a^2}{r^2}\right) \sin^2\theta \cos\theta \tag{1.3}$$

$$M_1 = \frac{\beta + {}^2/_5}{\beta + 1}, \qquad M_2 = \frac{\beta}{1+\beta}$$

$$\left(v_r = \frac{1}{r^2 \sin\theta} \frac{\partial\psi}{\partial\theta}, \quad v_\theta = -\frac{1}{r\sin\theta} \frac{\partial\psi}{\partial r}\right)$$

where a is the particle radius, v_r and v_θ are the fluid velocity components, and β is the ratio of viscosities of the fluid in- and outside the drop (for a solid particle $\beta = \infty$).

Let us determine the distribution of the diffusible substance (or heat) in the fluid and the diffusion stream on the surface of a spherical particle in the flow field (1.3) on the assumption that complete absorption of that substance, whose concentration away from the sphere is constant, takes place on the particle surface. Assuming that the Péclet number $P = \alpha a^2 / D \gg 1$ (D is the coefficient of diffusion), we can neglect the diffusive transfer of the substance along the particle surface, since it is much smaller than that normal to its surface.

The equation of steady convective diffusion in the boundary layer and the boundary conditions can be written in the form

$$v_r \frac{\partial c}{\partial r} + \frac{1}{r} v_\theta \frac{\partial c}{\partial\theta} = D \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial c}{\partial r}\right) \tag{1.4}$$

$$r = a, \quad c = 0; \quad r \to \infty, \quad c = c_0$$

where c is the concentration. Passing from variables $r\theta$ to $\psi\theta$, we reduce problem (1.4) to the following:

$$\frac{\partial c}{\partial\theta} = -D \sin\theta \frac{\partial}{\partial\psi}\left(r^2 \frac{\partial\psi}{\partial r} \frac{\partial c}{\partial\psi}\right) \tag{1.5}$$

$$\psi = 0, \quad c = 0; \quad \psi \to \infty, \quad c = c_0 \tag{1.6}$$

To complete the formulation of this problem in new variables we need one more condition, which can be derived on the following considerations. The stream of fluid along trajectories (which we shall call diffusion trajectories) originating at infinity and ending at certain points of the sphere surface has the highest concentration of the diffusible substance. Hence we must assume that the concentration along these trajectories must be equal to that at infinity, i.e., to c_0.

Let us consider the flow field (1.3). Obviously all diffusion trajectories are half-lines $\theta = \theta_0$, $a < r < \infty$, whose disposition depends on the sign of α; for $\alpha > 0$ the half-lines occupy the whole of the plane $\theta = \pi / 2$ outside the sphere; for $\alpha < 0$ there are only two diffusion trajectories, namely, the half lines $\theta = 0$ and $\theta = \pi$, $a < r < \infty$. Hence the additional conditions are

$$\theta = \theta_0, \qquad c = c_0 \qquad (r > a) \tag{1.7}$$

where θ_0 is to be chosen so that

$$\theta_0 = \pi / 2 \quad \text{for } \alpha > 0 \tag{1.8}$$

$$\theta_0 = \begin{cases} 0, & 0 \leqslant \theta \leqslant \pi / 2 \\ \pi, & \pi / 2 \leqslant \theta \leqslant \pi \end{cases} \quad \text{for } \alpha < 0 \tag{1.9}$$

It should be noted that (1.7) is a limit condition and must be satisfied for $P \to \infty$.

Expansion of $r^2 \partial\psi / \partial r$ in the sphere neighborhood into a series in ψ is used in the solution of problem (1.5) – (1.9), which makes it possible to reduce (1.5) to the equation of heat conduction. Since the principal terms of this expansion differ in the case of a solid particle from those of a drop, the two cases will be considered separately.

2. Diffusion on a solid sphere. In the neighborhood of the sphere surface the stream function (1.3) is of the form

$$\psi = {}^{15}/_2 \alpha a \, (r - a)^2 \sin^2 \theta \cos\theta + O\left((r - a)^3\right)$$

from which we obtain

$$r^2 \frac{\partial\psi}{\partial r} = 30^{1/2} a^{5/2} \sin\theta \, (\psi\alpha \cos\theta)^{1/2} \operatorname{sgn}(\alpha\cos\theta) + O(\psi) \tag{2.1}$$

Confining the expansion to its first term, we substitute (2.1) into (1.5) and introduce the new variable

$$t = 30^{1/2} D \, |\alpha|^{1/2} a^{5/2} A(\theta) \operatorname{sgn}\alpha \tag{2.2}$$

$$A(\theta) = \operatorname{sgn}(\cos\theta) \int_\theta^{\theta_0} \sin^2\theta \, |\cos\theta|^{1/2} d\theta \tag{2.3}$$

The problem (1.5) – (1.9) reduces now to the equation of heat conduction

$$\frac{\partial c}{\partial t} = D \frac{\partial}{\partial\psi}\left(\psi^{1/2} \frac{\partial c}{\partial\psi}\right)$$

with boundary conditions (1.6) and the initial condition $t = 0$ and $c = c_0$, which follows from (1.7), (2.2) and (2.3). The solution of this problem is of the form

$$c = c_0 \left(\frac{4}{9}\right)^{1/3} \Gamma^{-1}\left(\frac{4}{3}\right) \int_0^\eta \exp\left(-\frac{4}{9}\eta^3\right) d\eta \qquad \left(\eta = \frac{|\psi|^{1/2}}{t^{1/3}}\right) \tag{2.4}$$

Formula (2.4) with (2.2) and (2.3) defines the concentration distribution around the sphere. The diffusion flux on the sphere is

$$j = D\left(\frac{\partial c}{\partial r}\right)_{r=a} = c_0 \frac{(^5/_3)^{1/3}}{\Gamma(^1/_3)} \left(\frac{\alpha D^2}{a}\right)^{1/3} \sin\theta \, |\cos\theta|^{1/2} A^{-1/3}(\theta) \tag{2.5}$$

By virtue of (2.3), (1.8) and (1.9) the integral $A(\theta)$ can be expressed as follows:

$$A(\theta) = \begin{cases} \Omega(\theta), & 0 \leqslant \theta \leqslant \pi/2 \\ \Omega(\pi - \theta) & \pi/2 \leqslant \theta \leqslant \pi \end{cases} \quad \text{for } \alpha > 0 \tag{2.6}$$

$$A(\theta) = \begin{cases} \Omega(\theta) - \Omega(0), & 0 \leqslant \theta \leqslant \pi/2 \\ \Omega(\pi - \theta) - \Omega(0), & \pi/2 \leqslant \theta \leqslant \pi \end{cases} \quad \text{for } \alpha < 0 \tag{2.7}$$

Here

$$\Omega(\theta) = \int_\theta^{\pi/2} \sin^2\theta \cos^{1/2}\theta d\theta = \Omega(0) + \frac{2}{5}\sin\theta\cos^{3/2}\theta - \frac{4}{5}E\left(\frac{\theta}{2}, \sqrt{2}\right)$$

$$\Omega(0) = {}^1/_5 ({}^1/_2\pi)^{3/2}\,\Gamma^{-2}({}^5/_4) \tag{2.8}$$

where $E(\theta/2, \sqrt{2})$ is an elliptic integral of the second kind.

Thickness of the diffusion boundary layer is estimated by formula

$$\delta = Dc_0 / j \tag{2.9}$$

For $\theta \to 0$ and $\theta \to \pi$ (for $\alpha > 0$) and for $\theta \to \pi/2$ (for $\alpha < 0$) parameter $\delta \to \infty$. This means that compared to the sphere radius, the thickness of the diffusion boundary layer is not small in the neighborhood of the limit values of θ , and the method of solution used so far is inapplicable in this case. Equations (2.5) – (2.9) imply that these neighborhoods tend to decrease with decreasing Péclet number.

It should be noted that their effect on the total flux on the particle is negligible.

The total diffusion flux on the particle is determined by integrating (2.5) over its surface with the use of relationships (2.6) – (2.8). Calculations prove that the total flux is independent of the sign of α and is defined by

$$I = \frac{6\pi\,({}^5/_3)^{2/3}}{\Gamma({}^4/_3)}\,\Omega^{2/3}(0)\,c_0(|\alpha|D^2a^5)^{1/3} = ({}^9/_5)^{1/3}\,\pi^2\Gamma^{-1}({}^4/_3)\Gamma^{-4/3}({}^5/_4)\,c_0(|\alpha|D^2a^5)^{1/3} \approx$$

$$\approx 15.3c_0(|\alpha|D^2a^5)^{1/3} \tag{2.10}$$

3. Diffusion on a drop. By virtue of (1.3) in the vicinity of a spherical drop we have

$$r^2\frac{\partial\psi}{\partial r} = \frac{3}{\beta+1}\alpha a^4 \sin^2\theta\cos\theta + O(\psi) \tag{3.1}$$

By analogy to Sect. 2 we introduce the new variable

$$t = \frac{3}{\beta+1}D\alpha a^4 A(\theta), \qquad A(\theta) = \int_\theta^{\theta_0} \sin^3\theta\cos\theta\,d\theta \tag{3.2}$$

Equation (1.5) now reduces to the usual equation of heat conduction whose solution for boundary conditions (1.6) and the initial condition corresponding to (1.7) – (1.9) is of the form

$$c = c_0\,\frac{2}{\sqrt{\pi}}\int_0^\eta \exp(-\eta^2)\,d\eta \qquad \left(\eta = \frac{|\psi|}{2\sqrt{t}}\right) \tag{3.3}$$

Taking into consideration the definition of θ_0 in (1.7) – (1.9) and using (3.2), we obtain

$$4A(\theta) = \begin{cases} 1 - \sin^4\theta & \text{for } \alpha > 0 \\ -\sin^4\theta & \text{for } \alpha < 0 \end{cases} \tag{3.4}$$

The diffusion flux on the drop surface is

$$j = D\left(\frac{\partial c}{\partial r}\right)_{r=a} = c_0\left[\frac{3D\alpha}{\pi(1+\beta)A(\theta)}\right]^{1/2}\sin^2\theta\,|\cos\theta| \qquad (3.5)$$

It is clear that, unlike in the case of a solid sphere, the diffusion flux on a drop is independent of its size. Integrating (3.5) over the surface of the drop and taking into account (3.4) and (3.3), we find that the total flux on a drop is independent of the sign of α and is given by

$$I = 4\sqrt{3\pi}\left(\frac{D|\alpha|}{1+\beta}\right)^{1/2} c_0 a^2 \qquad (3.6)$$

The flow past a particle considered here, as well as the flow past of straight-line stream, can be observed only in certain particular pattern of particle motion. An example of such flow is provided by the field of laminar flow past a completely entrained by the stream particle moving along the axis of a diffuser (or convergent nozzle). Formulas (2.10) and (3.6) may be used as input equations for deriving solutions for more complex flows. It should be noted that a shear flow of the kind defined by (1.1) may, in the case of an arbitrary tensor α be represented as the sum of three tensors of the kind (1.2) by rotating the axes of coordinates.

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